Polynomials associated with equilibria of affine Toda–Sutherland systems

2004 J. Phys. A: Math. Gen. 37 11401

(http://iopscience.iop.org/0305-4470/37/47/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 11401–11406 PII: S0305-4470(04)84501-3

Polynomials associated with equilibria of affine Toda–Sutherland systems

S Odake[1] **and R Sasaki**[2]

[1] Department of Physics, Shinshu University, Matsumoto 390-8621, Japan
[2] Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Received 3 August 2004, in final form 11 October 2004
Published 10 November 2004
Online at stacks.iop.org/JPhysA/37/11401
doi:10.1088/0305-4470/37/47/009

Abstract
An affine Toda–Sutherland system is a *quasi-exactly solvable* multi-particle dynamics based on an affine simple root system. It is a 'cross' between two well-known integrable multi-particle dynamics, an affine Toda molecule (exponential potential, periodic nearest-neighbour interaction) and a Sutherland system (inverse sine-square interaction). Polynomials describing the equilibrium positions of affine Toda–Sutherland systems are determined for all affine simple root systems.

PACS numbers: 02.20.−a, 02.30.Gp, 02.30.Ik

1. Introduction

Given a multi-particle dynamical system, to find and describe its equilibrium position has practical as well as theoretical significance. As is well known, near the equilibrium the system is reduced to a collection of harmonic oscillators and their spectra give the exact order $\hbar$ part of the full quantum spectra [1]. Naively, one could describe the equilibrium position by zeros of a certain polynomial. In this way one obtains the celebrated classical orthogonal polynomials for *exactly solvable* multi-particle dynamics. For the Calogero systems [2] based on the A and B (C, BC and D) root systems, the equilibrium positions correspond to the zeros of the Hermite and Laguerre polynomials [3–6]. For the Sutherland systems [7] based on the A and B (C, BC and D) root systems, the equilibrium positions correspond to the zeros of the Chebyshev and Jacobi polynomials [6]. Polynomials describing the equilibria of the Calogero and Sutherland systems based on the exceptional root systems are also determined [8]. In all these cases the frequencies of small oscillations at the equilibrium are '*quantized*' [6, 9]. For another family of multi-particle dynamics based on root systems, the Ruijsenaars–Schneider systems [10], which are *deformation* of the Calogero and Sutherland systems, the corresponding polynomials are determined [11, 12]. They turn out to be *deformations* of the Hermite, Laguerre and Jacobi polynomials which inherit the orthogonality [12]. The frequencies of small oscillations at the equilibrium are also '*quantized*' [11]. Another interesting feature is that the equations determining the equilibrium look like *Bethe ansatz* equations.

One is naturally led to a similar investigation for partially solvable or *quasi-exactly solvable* [13] multi-particle dynamics. From a not-so-long list of known quasi-exactly solvable multi-particle dynamical systems [14], we choose the so-called affine Toda–Sutherland systems [15] and determine polynomials describing the equilibrium positions. These polynomials, as well as all the polynomials mentioned above, are characterized as having *integer* coefficients only.

2. Affine Toda–Sutherland systems

The affine Toda–Sutherland systems are quasi-exactly solvable [13] multi-particle dynamics based on any crystallographic root system. Roughly speaking, they are obtained by 'crossing' two well-known integrable dynamics, the affine-Toda molecule (exponential potential, periodic nearest-neighbour interaction) and the Sutherland system (inverse sine-square interaction). Given a set of *affine simple roots* $\Pi_0 = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$, $\alpha_j \in \mathbb{R}^r$, let us introduce a *prepotential* W [16]

$$W(q) = g\sum_{j=0}^{r} n_j \log|\sin(\alpha_j q)|, \qquad q = {}^t(q_1, \ldots, q_r) \in \mathbb{R}^r, \tag{1}$$

where g is a positive coupling constant and $\{n_j\}$ are the Dynkin–Kac labels for Π_0. That is, they are the integer coefficients of the affine simple root α_0; $-\alpha_0 = \sum_{j=1}^{r} n_j\alpha_j$, $n_0 \equiv 1$. For simply-laced and un-twisted non-simply laced affine root systems α_0 is the lowest long root, whereas for twisted non-simply laced affine root systems, α_0 is the lowest short root. In either case $h \stackrel{\text{def}}{=} \sum_{j=0}^{r} n_j$ is the *Coxeter number*. This leads to the classical Hamiltonian

$$H_C = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W(q)}{\partial q_j}\right)^2. \tag{2}$$

It is shown [15] that the principal equilibrium position $\bar{q}$ is given by a *universal* formula in terms of the dual Weyl vector $\varrho^\vee$:

$$\frac{\partial W(\bar{q})}{\partial q_j} = 0 \quad \Leftrightarrow \quad \bar{q} = \frac{\pi}{h}\varrho^\vee, \qquad \varrho^\vee \stackrel{\text{def}}{=} \sum_{j=1}^{r}\lambda_j^\vee. \tag{3}$$

The dual fundamental weight $\lambda_j^\vee$ is defined in terms of the fundamental weight λ_j by $\lambda_j^\vee \stackrel{\text{def}}{=} (2/\alpha_j^2)\lambda_j$, which satisfies $\alpha_j \cdot \lambda_k^\vee = \delta_{jk}$. At the equilibrium, the classical multi-particle dynamical system (2) is reduced to a set of harmonic oscillators. The frequencies (not frequencies squared) of small oscillations at the equilibrium of the affine Toda–Sutherland model are given up to the coupling constant g by [15]

$$\frac{1}{\sin^2\frac{\pi}{h}}\left\{m_1^2, m_2^2, \ldots, m_r^2\right\},$$

where m_j^2 are the so-called affine Toda masses [17]. Namely, they are the eigenvalues of a symmetric $r \times r$ matrix M, $M_{kl} = \sum_{j=0}^{r} n_j(\alpha_j)_k(\alpha_j)_l$, or $M = \sum_{j=0}^{r} n_j\alpha_j \otimes \alpha_j$, which encode the integrability of affine Toda field theory. In [17] it is shown for the non-twisted cases that the vector $\mathbf{m} = {}^t(m_1, \ldots, m_r)$, if ordered properly, is the *Perron–Frobenius* eigenvector of the incidence matrix (the Cartan matrix) of the corresponding root system.

The corresponding *quantum* Hamiltonian (with $\hbar = 1$) [1, 16] is

$$H_Q = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W(q)}{\partial q_j}\right)^2 + \frac{\partial^2 W(q)}{\partial q_j^2}\right], \tag{4}$$

which is partially solvable or *quasi-exactly solvable* for some affine simple root systems. Namely for $A_{r-1}^{(1)}$, $D_3^{(1)}$, $D_{r+1}^{(2)}$, $C_r^{(1)}$ and $A_{2r}^{(2)}$, the above Hamiltonian (4) is known to have a few exact eigenvalues and corresponding exact eigenfunctions [15].

The polynomials related to the equilibrium position $\bar{q}$ are easy to define for the classical root systems, A, B, C and D. As in the Sutherland cases, we introduce a polynomial having zeros at $\{\sin\bar{q}_j\}$ or $\{\cos 2\bar{q}_j\}$:

$$P_r(q) \propto \prod_{j=1}^{r}(x - \sin\bar{q}_j), \qquad \prod_{j=1}^{r}(x - \cos 2\bar{q}_j). \tag{5}$$

For the exceptional root systems, let us choose a set of D vectors $\mathcal{R}$

$$\mathcal{R} = \{\mu^{(1)}, \ldots, \mu^{(D)} | \mu^{(a)} \in \mathbb{R}^r\},$$

which form a single orbit of the corresponding Weyl group. For example, they are the set of roots Δ itself for simply laced root systems, the set of long (short, middle) roots $\Delta_L(\Delta_S, \Delta_M)$ for non-simply laced root systems and the so-called sets of *minimal weights*. The latter is better specified by the corresponding fundamental representations, which are all the fundamental representations of A_r, the vector (**V**), spinor (**S**) and conjugate spinor ($\bar{\mathbf{S}}$) representations of D_r and 27 ($\overline{27}$) of E_6 and 56 of E_7. By generalizing the above examples (5), we define polynomials

$$P_\Delta^{\mathcal{R}}(x) \propto \prod_{\mu\in\mathcal{R}}(x - \sin(\mu\bar{q})), \qquad \prod_{\mu\in\mathcal{R}}(x - \cos(2\mu\bar{q})). \tag{6}$$

Therefore, in most cases, determination of the coefficients of the polynomials is reduced to the evaluation of the elementary symmetric functions of $\sin(\mu_j\bar{q})$ or $\cos(2\mu_j\bar{q})$, $\mu_j \in \mathcal{R}$. This can be easily achieved by hand and with the help of formula manipulation software. For a more general treatment we refer to our previous paper [8].

The resulting polynomials for various affine root systems Π_0 are (we follow the affine Lie algebra notation used in [15, 17]):

$A_{r-1}^{(1)}$. In this case the equilibrium position is exactly the same as that of the A_{r-1} Sutherland [7] and A_{r-1} Ruijsenaars–Sutherland system [12], $\bar{q} = (\pi/2h)^t(r-1, r-3, \ldots, -(r-1))$ with $h = r$. Thus the polynomial is also the same, the Chebyshev polynomial of the first kind: $2^{r-1}\prod_{j=1}^{r}(x - \sin\bar{q}_j) = T_r(x) = \cos r\varphi$, $x = \cos\varphi$.

$B_r^{(1)}$ and $D_{r+1}^{(2)}$ & $A_{2r}^{(2)}$. The Coxeter number is $h = 2r$ for $B_r^{(1)}$, $h = r+1$ for $D_{r+1}^{(2)}$ and $h = 2r+1$ for $A_{2r}^{(2)}$. The equilibrium position is equally spaced by $\bar{q} = (\pi/h)^t(r, r-1, \ldots, 1)$. We obtain the Chebyshev polynomial of the second kind, $U_n(x) = \sin(n+1)\varphi/\sin\varphi$, $x = \cos\varphi$, for $B_r^{(1)}$ and a product of them for $D_{r+1}^{(2)}$ and a sum of them for $A_{2r}^{(2)}$,

$$2^{r-1}\prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = \begin{cases} (x+1)U_{r-1}(x), & B_r^{(1)}, \\ (x+1)U_{r/2}(x)U_{(r-2)/2}(x) + 1/2, & D_{r+1}^{(2)}, \quad r: \text{even}, \\ (x+1)U_{(r-1)/2}(x)^2, & D_{r+1}^{(2)}, \quad r: \text{odd}, \\ (U_r(x) + U_{r-1}(x))/2, & A_{2r}^{(2)}. \end{cases} \tag{7}$$

$C_r^{(1)}$ and $A_{2r-1}^{(2)}$. The Coxeter number is $h = 2r$ for $C_r^{(1)}$ and $h = 2r-1$ for $A_{2r-1}^{(2)}$. The equilibrium position is equally spaced by $\bar{q} = (\pi/2h)^t(2r-1, 2r-3, \ldots, 3, 1)$. We obtain the Chebyshev polynomial of the first kind $T_r(x)$ for $C_r^{(1)}$ and a sum of them for $A_{2r-1}^{(2)}$,

$$2^{r-1}\prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = \begin{cases} T_r(x), & C_r^{(1)}, \\ T_r(x) + T_{r-1}(x), & A_{2r-1}^{(2)}. \end{cases} \tag{8}$$

$D_r^{(1)}$. The Coxeter number is $h = 2(r-1)$ and the equilibrium position is equally spaced by $\bar{q} = (\pi/h)^t(r-1, r-2, \ldots, 1, 0)$. We obtain the Chebyshev polynomial of the second kind

$$2^{r-2}\prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = (x^2-1)U_{r-2}(x). \tag{9}$$

$E_6^{(1)}$. The Coxeter number is $h = 12$ and the equilibrium position is not equally spaced by $\bar{q} = (\pi/h)^t(4\sqrt{3}, 4, 3, 2, 1, 0)$. We consider the set of minimal weights 27 and the set of positive roots Δ_+, which consists of 36 roots. The polynomials are

$$2^{20}\prod_{\mu\in 27}(x - \sin(\mu\bar{q})) = (-1+x)x^3(1+x)(-1+2x)^2 \times (1+2x)^2(-1+2x^2)^2(-3+4x^2)^3(1-16x^2+16x^4)^2, \tag{10}$$

$$2^{27}\prod_{\mu\in\Delta_+}(x - \cos(2\mu\bar{q})) = x^6(1+x)^3(-1+2x)^6(1+2x)^7(-3+4x^2)^7. \tag{11}$$

$E_7^{(1)}$. The Coxeter number is $h = 18$ and the equilibrium position is not equally spaced by $\bar{q} = (\pi/2h)^t(17\sqrt{2}, 10, 8, 6, 4, 2, 0)$. We consider the set of minimal weights 56 and the set of positive roots Δ_+, which consists of 63 roots. The 56 is even, i.e. if $\mu \in 56$ then $-\mu \in 56$. The positive part of 56 is denoted by 56_+. The polynomials are

$$2^{24}\prod_{\mu\in 56_+}(x - \cos(2\mu\bar{q})) = x^4(-3+4x^2)^3(-3+36x^2-96x^4+64x^6)^3, \tag{12}$$

$$2^{59}\prod_{\mu\in\Delta_+}(x - \cos(2\mu\bar{q})) = (1+x)^4(-1+2x)^7(1+2x)^7(-1+6x+8x^3)^8(1-6x+8x^3)^7. \tag{13}$$

$E_8^{(1)}$. The Coxeter number is $h = 30$ and the equilibrium position is not equally spaced by $\bar{q} = (\pi/h)^t(23, 6, 5, 4, 3, 2, 1, 0)$. We consider the set of positive roots Δ_+, which consists of 120 roots. The polynomial is

$$2^{116}\prod_{\mu\in\Delta_+}(x - \cos(2\mu\bar{q})) = (1+x)^4(-1+2x)^8(1+2x)^8(-1-2x+4x^2)^8(-1+2x+4x^2)^8 \times (1+8x-16x^2-8x^3+16x^4)^8(1-8x-16x^2+8x^3+16x^4)^9. \tag{14}$$

$F_4^{(1)}$ and $E_6^{(2)}$. The Coxeter number is $h = 12$ for $F_4^{(1)}$ and $h = 9$ for $E_6^{(2)}$ and the equilibrium position is not equally spaced by $\bar{q} = (\pi/h)^t(8, 3, 2, 1)$. We consider the set of long positive roots Δ_{L+} and short positive roots Δ_{S+}, both of which consist of 12 roots reflecting the self-duality of F_4 Dynkin diagram. The polynomials for $F_4^{(1)}$ are

$$2^9\prod_{\mu\in\Delta_{S+}}(x - \cos(2\mu\bar{q})) = x^2(1+x)(-1+2x)^2(1+2x)^3(-3+4x^2)^2, \tag{15}$$

$$2^9 \prod_{\mu\in\Delta_{L+}} (x - \cos(2\mu\bar{q})) = x^2(1+x)(-1+2x)^2(1+2x)(-3+4x^2)^3. \tag{16}$$

The polynomials associated with the twisted affine root system $E_6^{(2)}$ are

$$2^{12} \prod_{\mu\in\Delta_{S+}} (x - \cos(2\mu\bar{q})) = (1+2x)^3(1-6x+8x^3)^3, \tag{17}$$

$$2^{12} \prod_{\mu\in\Delta_{L+}} (x - \cos(2\mu\bar{q})) = 2(-1+x)(1+2x)^2(1-6x+8x^3)^3. \tag{18}$$

$G_2^{(1)}$ and $D_4^{(3)}$. The Coxeter number is $h = 6$ for $G_2^{(1)}$ and $h = 4$ for $D_4^{(3)}$ and the equilibrium position is $\bar{q} = (\pi/2h)^t(3\sqrt{6}, \sqrt{2})$. We consider the set of long positive roots Δ_{L+} and short positive roots Δ_{S+}, both of which consists of three roots reflecting the self-duality of G_2 Dynkin diagram. The polynomials for the untwisted $G_2^{(1)}$ are

$$2^3 \prod_{\mu\in\Delta_{S+}} (x - \cos(2\mu\bar{q})) = 2(1+x)(-1+2x)(1+2x), \tag{19}$$

$$2^3 \prod_{\mu\in\Delta_{L+}} (x - \cos(2\mu\bar{q})) = (-1+2x)^2(1+2x). \tag{20}$$

The polynomials for the twisted $D_4^{(3)}$ are

$$\prod_{\mu\in\Delta_{S+}} (x - \cos(2\mu\bar{q})) = x^2(1+x), \tag{21}$$

$$\prod_{\mu\in\Delta_{L+}} (x - \cos(2\mu\bar{q})) = x^2(-1+x). \tag{22}$$

Before closing this paper, let us briefly remark on the identities arising from *foldings* of root systems. Among them those relating two untwisted root systems, i.e. with superscript (1) are quite simple.

Folding $A_{2r-1}^{(1)} \to C_r^{(1)}$. The vector weights of A_{2r-1} ($2r$ dim.) become those of C_r ($2r$ dim.). This relates T_{2r} to T_r in (8) as

$$A_{2r-1}: \quad T_{2r}(x) = (-1)^r T_r(1-2x^2), \qquad C_r^{(1)}. \tag{23}$$

Folding $D_{r+1}^{(1)} \to B_r^{(1)}$. This gives a quite obvious relation as seen from (9) and (7).

Folding $E_6^{(1)} \to F_4^{(1)}$. In this folding the minimal weights 27 of E_6 become Δ_S (24 dim.) of F_4 plus three zero weights. Thus we obtain

$$E_6^{(1)}: \quad 2(10)/x^3 = (15)_{x\to 1-2x^2}, \qquad F_4^{(1)}. \tag{24}$$

We also obtain

$$E_6^{(1)}: \quad (11) = (15)^2 \times (16), \qquad F_4^{(1)}, \tag{25}$$

since the 72 roots of E_6 are decomposed into $2\Delta_S + \Delta_L$ (24 dim.) of F_4.

Folding $D_4^{(1)} \to G_2^{(1)}$. The vector weights of D_4 (8 dim.) decompose into Δ_S (6 dim.) plus two zero weights of G_2 leading to the identity

$$D_4^{(1)}: \quad 2(9)_{r=4}/(x-1) = (19), \qquad G_2^{(1)}. \tag{26}$$

Acknowledgment

S O and R S are supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, no 13135205 and no 14540259, respectively.

References

[1] Loris I and Sasaki R 2004 Quantum vs classical mechanics, role of elementary excitations *Phys. Lett.* A **327** 152–7

[2] Calogero F 1971 Solution of the one-dimensional *N*-body problem with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419–36

[3] Stieltjes T 1885 Sur quelques théorèmes d'Algèbre *Compt. Rend.* **100** 439–40
Stieltjes T 1885 Sur les polynômes de Jacobi *Compt. Rend.* **100** 620–2

[4] Szegö G 1939 *Orthogonal Polynomials* (New York: American Mathematical Society)

[5] Calogero F 1977 On the zeros of the classical polynomials *Lett. Nuovo Cimento* **19** 505–7
Calogero F 1977 Equilibrium configuration of one-dimensional many-body problems with quadratic and inverse quadratic pair potentials *Lett. Nuovo Cimento* **22** 251–3

[6] Corrigan E and Sasaki R 2002 Quantum vs classical integrability in Calogero–Moser systems *J. Phys. A: Math. Gen.* **35** 7017–62

[7] Sutherland B 1972 Exact results for a quantum many-body problem in one-dimension. II *Phys. Rev.* A **5** 1372–6

[8] Odake S and Sasaki R 2002 Polynomials Associated with Equilibrium Positions in Calogero–Moser systems *J. Phys. A: Math. Gen.* **35** 8283–314

[9] Loris I and Sasaki R 2004 Quantum and Classical Eigenfunctions in Calogero and Sutherland Systems *J. Phys. A: Math. Gen.* **37** 211–37

[10] Ruijsenaars S N and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys.* **170** 370–405
Ruijsenaars S N 1987 Complete integrability of relativistic Calogero–Moser systems and elliptic function identities *Commun. Math. Phys.* **110** 191–213

[11] Ragnisco O and Sasaki R 2004 Quantum vs classical integrability in Ruijsenaars–Schneider systems *J. Phys. A: Math. Gen.* **37** 469–79

[12] Odake S and Sasaki R 2004 Equilibria of 'discrete' integrable systems and deformations of classical polynomials *J. Phys. A: Math. Gen.* **37** at press (*Preprint* hep-th/0407155)

[13] See for example: Turbiner A V 1988 Quasi-exactly-soluble problems and sl(2,R) algebra *Commun. Math. Phys.* **118** 467–74
Ushveridze A G 1994 *Quasi-Exactly Solvable Models in Quantum Mechanics* (Bristol: Institute of Physics Publishing)

[14] Sasaki R and Takasaki K 2001 Quantum Inozemtsev model, quasi-exact solvability and $\mathcal{N}$-fold supersymmetry *J. Phys. A: Math. Gen.* A **34** 9533–54

[15] Khare A, Loris I and Sasaki R 2004 Affine Toda-Sutherland systems *J. Phys. A: Math. Gen.* A **37** 1665–80

[16] Bordner A J, Manton N S and Sasaki R 2000 Calogero–Moser models V: supersymmetry and quantum lax pair *Prog. Theor. Phys.* **103** 463–87
Khastgir S P, Pocklington A J and Sasaki R 2000 Quantum Calogero–Moser models: integrability for all root systems *J. Phys. A: Math. Gen.* **33** 9033–64

[17] Braden H W, Corrigan E, Dorey P E and Sasaki R 1990 Affine Toda field theory and exact S-matrices *Nucl. Phys.* B **338** 689–746